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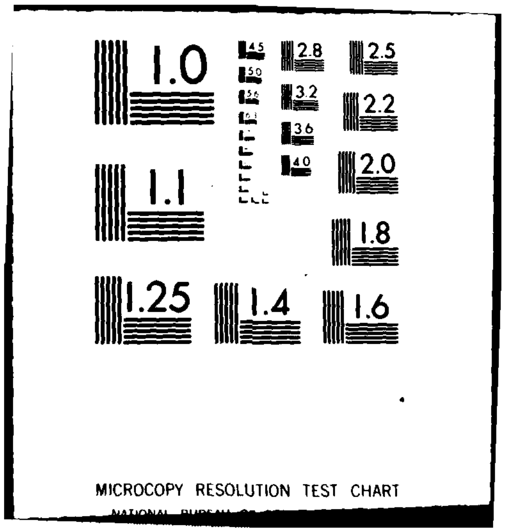
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FINAL SCIENTIFIC REPORT

ON

TWO-WAY ANALYSIS OF VARIANCE FOR WEIBULL POPULATIONS

AUGUST 1980

J. I. MC COOL

SKF REPORT NO.: AT80D048

SUBMITTED TO:

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20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

Expressions are given for the maximum likelihood estimators of the Weibull shape parameter using data from a two-way classification experiment under five hypothesized relationships for the scale parameter in each cell, viz, the scale parameter (1) is constant (2) has a multiplicative row effect (3) has a multiplicative column effect (4) has a multiplicative row and column effect, and (5) has multiplicative row and column and interaction effects. Procedures for analyzing a two-way layout without interaction (a randomized block design) are discussed and illisutrated. Techniques are given for estimation of, and

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inference on, the location parameter of the three-parameter Weibull distribution and a military application of the technique is given.

Software improvements are cited in the Monte Carlo simulation computer program for the computation of the distributions of pivotal functions required for analyzing Weibull regression experiments.

In these experiments, the response variable follows a two-paramter Weibull distribution with a scale parameter that varies inversely with a power of an external variable termed a "stress." WIth the software permit (1) testing the adequacy of the power function fit, (2) setting confidence limits on the shape parameter and stresslife exponent and (3) setting confidence limits on a specified percentile of the distribution at seven prescribed "stress" levels. Tables have been generated for the analysis of experiments conducted at two and three stress levels, with sample sizes ranging from 5 to 20.

Methodology for the analysis of a single classification experiment with Weibull distributed response is described and the contents of an applications oriented paper is described. Other applications of the methodolgy are cited.

Software is described for simulation of the distribution of the maximum likelihood estimates of the Weibull parameters that result when the data are grouped. Results of varying interval width and sample size are given.

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1.0 SINGLE CLASSIFICATION EXPERIMENTS

A paper containing a comprehensive treatment of sets of two-parameter Weibull data arising from single factor experiments, was published in the Journal of Statistical Planning and Inference (Vol. 3, 1979, pp. 39 - 68). A second paper, aimed primarily at an engineering audience was prepared and submitted to the American Society For Testing and Materials (ASTM) for presentation at their International Symposium on Contact Rolling Fatigue Testing of Bearing Steels to be held in May 1981. The paper will appear in the Proceedings of that meeting, which is planned to appear as an ASTM special technical publication.

This paper extends the range of the tabular values needed in conducting the analysis from a maximum sample size of $n = 10$ to a maximum of $n = 30$. Specifically, new tables are included for $n = 15(5)30$ with censoring amounts ranging from $r = 5(5)n$ and with $k = 2(1)5$ samples per experiment. In the paper, two illustrative numerical examples are used to illustrate each of the analysis procedures viz.:

(1) Testing the equality of shape parameters
(2) Testing the equality of scale parameters
(3) Setting confidence limits on the shape parameter
(4) Setting confidence limits on the tenth percentile of each population sampled, and
(5) Conducting a single range multiple comparison test to divide the population into groups.

The data used for illustration were the results of rolling contact fatigue tests conducted with several types of steel using two distinct types of tester. The data were collected under an Air Force sponsored program with one set of tests conducted at Wright-Patterson Air Force Base and the other at the Pratt and Whitney Division of United Technology, Inc.

The analysis shows that the life ranking of the steels when analyzed as a set was the same for both test devices. This methodology is now routinely applied within SKF to analyze data taken for industrial and DOD sponsors as well as for corporate use.

An application of the analysis to a study of the comparative effect of six bearing greases on the life of automotive wheel bearings, was conducted under U. S. Army Contract No. DAAK70-77-C-0034 and appears in the final report submitted to the U. S. Army Mobility Equipment Research and Development Command at Fort Belvoir, Virginia. The report is SKF Number AL78T022 entitled "Performance of Automotive Wheel Bearing Greases."

2.0 LOCATION PARAMETER ESTIMATION AND INFERENCE

During this contract year, work on inference for the Weibull location parameter was extended, applied, and reported in full.

Extensions comprised (1) the computation of additional critical values of the statistic used for testing whether the Weibull location parameter exceeds zero. Values are now available for sample sizes ranging from 10 to 40. (2) Additional computations were performed for the purpose of comparing the power of this inferential technique to the Mann-Fertig method. The power was found to be almost identical. (3) An analytical proof was developed of the decreasing monotonicity with λ of the ratio $w(\lambda) = \hat{\beta}(r_1)/\hat{\beta}(r_2)$, where $r_1 < r_2$ and where $\hat{\beta}(r)$ is the ML estimator of the Weibull shape parameter based on the first and r-th order statistics when an amount λ is subtracted from each order statistic.

The more general result regarding the monotonicity of $w(\lambda)$ when $\hat{\beta}(r)$ is defined as the ML estimator based on all order statistics has not been proven but has repeatedly been demonstrated with data samples.

Two applications of the methodology were made using actual data samples taken from two diverse areas of activity. The first application was to a data sample that represented time to ignition events for fuzing devices. The possible application of the three parameter Weibull model in this context had been suggested in private communication with Dr. B. Kurkjian when he was chief mathematician of the U. S. Army Materiel Command. The model is attractive for this application because if it fits, the location parameter will correspond to the "safe" time prior to which an armed device cannot undergo ignition. Numerous contacts were made at Wright-Patterson (W. Romans Air Force Logistics), at Kirtland (Neal Chamblee), and at Eglin Air Force Bases to discover sources of time-to-event data. After discussions with staff members of the engineering and quality assurance departments, Mr. Jasper Glover, head of the Reliability Department at Eglin, finally referred us to Mr. Charles Yates of his department, who kindly supplied a sample of 29 time-to-event data unidentified as to source. The data were analyzed and showed that a mixture model was a better fit to the data than the three parameter Weibull model. The results of the analysis were presented in a Technical Report, "Analysis of Time-To-Event Data Supplied by Eglin Air Force Base." A copy is included herein as Appendix I. Copies of the report were sent to Mr. Yates at Eglin and on his recommendation, to Mr. T. Mitchell who is responsible for the setting of safety requirements. The results were also discussed with, and a copy of the report sent to a Mr. L. Cox of the Army's Harry Diamond Laboratories. Mr. Cox is concerned with the performance of fuzing devices developed by the Army.

The Eglin analysis taught that the methodology is useful for distinguishing two superficially similar looking types of data, namely: samples from a mixture of a pair of 2-parameter

Weibull population. To make this distinction, one censors a proportion of the upper end of the data and reperforms the hypothesis test. If a significant result is no longer so after the censoring, the mixture model is assumed to obtain.

A second application was made using data taken from the results of a foot race. The object was to determine whether a bound on human performance potential could be found. Again, it was found that a Weibull mixture model was in better accord with the data than the three parameter Weibull model.

The extended tabular data and two illustrative examples discussed above have been described in a paper, "Inference on the Weibull Location Parameter," submitted for publication to Technometrics. A verbal presentation having the same title will be made at the Joint Meetings of the American Statistical Association to be held in Houston, Texas in August 1980. A handout synopsizing the talk has been prepared and is included herein as Appendix II.

3.0 WEIBULL REGRESSION OR ACCELERATED TEST ANALYSIS

During a previous contract year, we developed methodology for drawing exact inferences in this setting: (1) Type II censored life tests are conducted at various levels of a factor referred to as a stress. (2) At each stress level, the life follows a two parameter Weibull distribution with a shape parameter β that is invariant with stress and a scale parameter that varies inversely with a power γ of the stress. Using the distribution of certain pivotal functions determined by Monte Carlo sampling, it was found possible to set confidence limits on:

(1) The exponent in the relation between scale parameter and stress,

(2) The Weibull shape parameter,

(3) A percentile of the life distribution at any specific stress.

A verbal presentation of this material was made at the ASA joint conferences in August 1979. A copy of the handout material distributed at the conference is given in Appendix III.

A paper was prepared describing this work and illustrating the methodology on four rolling contact fatigue test samples conducted at four stress levels. Editorial changes to the paper were made in the current contract year and the paper, entitled "Confidence Limits for Weibull Regression with Censored Data," appeared in the IEEE Transactions on Reliability, Vol. R-29, No. 2, June 1980.

A question raised by a referee regarding the goodness of the power function model for stress-life, prompted the recognition that one could perform such a test using the ratio of $\hat{\beta}(1)$, the ML shape parameter estimate unconstrained by any relation among the scale parameters, to the estimate $\hat{\beta}$ under the power function constraint. Accordingly appropriate code was added to the simulation program REGSIM to calculate $\hat{\beta}(1)$.

The program REGSIM as originally configured, calculates the distribution of five random variables; these variables being specified functions of the ML estimates $\hat{\beta}$, and the p-th quantile $\hat{x}_p(S)$ estimated at stress S.

The program REGSIM has now been modified to accommodate the calculation of five additional random variables giving a total of ten. The first three are as follows:

(1) $\hat{\beta} / \beta$

(2) $\hat{\beta}(1)/\hat{\beta}$

(3) $(\hat{\gamma}-\gamma) \cdot \hat{\beta}$

The next "k" are the values of the random variable $\hat{\beta}\ln(\hat{x}_p/x_p)$ computed at the k stresses, at which life tests are performed. The next (7-k) random variables are the values of $\hat{\beta} \cdot \ln(\hat{x}_p/x_p)$ at (7-k), other specified stresses.

The distribution of the first random variable above, $\hat{\beta}/\beta$, is needed for setting confidence limits on β. The second is used for testing the adequacy of the power function fit. The distribution of the third random variable is needed for calculating confidence intervals on the stress-life exponent. Finally the distribution of $\hat{\beta}\ln(\hat{x}_p/x_p)$ is used for setting confidence intervals on x at any given stress level.

Computer runs have been made for k = 2, with life tests at stresses $s_1 = 1.0$ and $s_2 = 1.2$. Supplementary stresses were taken at 0.5 (0.1) 0.9. For k = 3, the life tests were presumed to be run at S = 1.0, 1.1, and 1.2 with supplementary stresses of 0.6(0.1)0.9.

The percentile p = 0.10 was used throughout and the sample sizes used for each life test were n = r = 5, 10, 15, 20. Inasmuch as these distributions are invariant with respect to the scale of the stress variable, they apply when the stresses, in whatever physical units they are expressed, are proportional to the values used in the simulation runs.

A short paper will be prepared, aimed at a user audience, presenting the tables and illustrating their use.

4.0 TWO-WAY CLASSIFICATION EXPERIMENTS

The likelihood equations for a general two-way factorial analysis with Weibull response have been derived. There are presumed to be "a" rows and "b" columns, representing the levels of factors A and B respectively. Five separate hypotheses have been considered for the scale parameter η_{ij}, applicable when sampling row "i" and column "j":

$$H_1: \quad \eta_{ij} = a_i \, b_j \, c_{ij} \, \eta$$

$$H_2: \quad \eta_{ij} = a_i \, b_i \, \eta$$

$$H_3: \quad \eta_{ij} = a_i \, \eta$$

$$H_4: \quad \eta_{ij} = b_j \, \eta$$

$$H_5: \quad \eta_{ij} = \eta$$

a_i, b_j and c_{ij} are multiplicative row, column and interaction effects subject to the constraints

$$\prod_{i=1}^{a} a_i = \prod_{j=2}^{b} b_j = \prod_{i=1}^{a} c_{ij} = \prod_{j=2}^{b} c_{ij} = 1 \, .$$

η is a constant "base level" scale parameter value. H_1 is the least restrictive hypothesis. Under H_1 each cell of the data layout has its own unique scale parameter value. H_5 is the most restrictive hypothesis under which all cells are presumed to have the same scale parameter. Under H_2 there is a row and column effect, but no interaction. Under H_3 there is only a row, and under H_4, only a column effect.

The estimates of β and η_{ij} which maximize the likelihood function under each hypothesis subject to the row and column constraints are listed in Table 1. They represent the case where n items are tested in each cell until the first r fail. We define $x_{ij(k)}$ as the k-th ordered life within cell (i,j).

HYPOTHESIS	ML SHAPE PARAMETER ESTIMATE Found by Solving	EQ. FOR $\hat{\eta}_{ij}^{\hat{\beta}}$
$H_1 : \eta_{ij} = a_i b_j c_{ij} \eta$	$1/\hat{\beta}_1 + S_{..}/abr - \frac{1}{ab} \sum_{i=1}^{a} \sum_{j=1}^{b} T_{ij}/V_{ij} = 0$	V_{ij}/r
$H_2 : \eta_{ij} = a_i b_j \eta$	$1/\hat{\beta}_2 [1 + K^{-1} \sum_{i=1}^{a} \sum_{j=1}^{b} \ln [\frac{V_{i.} V_{.j}}{V_{..}^2}] \cdot (\frac{V_{ij}}{V_{i.} V_{.j}})] -$	$K V_{i.} V_{.j}/abr$
	$K^{-1} \sum_{i=1}^{a} \sum_{j=1}^{b} T_{ij}/V_{i.}/V_{.j} = 0$ $(K \equiv \sum_{i=1}^{a} \sum_{j=1}^{b} [V_{ij}/V_{i.} V_{.j}])$	
$H_3 : \eta_{ij} = a_i \eta$	$1/\hat{\beta}_3 + S_{..}/abr - a^{-1} \sum_{i=1}^{a} (\sum_{j=1}^{b} T_{ij} / \sum_{j=1}^{b} V_{ij}) = 0$	$\sum_{j=1}^{b} V_{ij}/br$
$H_4 : \eta_{ij} = b_j \eta$	$1/\hat{\beta}_4 + S_{..}/abr - b^{-1} \sum_{j=1}^{b} (\sum_{i=1}^{a} T_{ij} / \sum_{i=1}^{a} V_{ij}) = 0$	$\sum_{i=1}^{a} V_{ij}/ar$
$H_5 : \eta_{ij} = \eta$	$1/\hat{\beta}_5 + S_{..}/abr - \sum_{i=1}^{a} \sum_{j=1}^{b} T_{ij} / \sum_{i=1}^{a} \sum_{j=1}^{b} V_{ij} = 0$	$\sum_{i=1}^{a} \sum_{j=1}^{b} V_{ij}/abr$

TABLE 1. ML ESTIMATION EQUATIONS FOR FACTORIAL EXPERIMENTS UNDER VARIOUS HYPOTHESES

Define:

$$S_{ij} = \sum_{k=1}^{r} \ln [x_{ij(k)}]$$

$$S_{..} = \sum_{j=1}^{a} \sum_{i=1}^{b} S_{ij}$$

$$T_{ij} = \sum_{k=1}^{n} x_{ij(k)}^{\hat{\beta}} \ln x_{ij(k)}$$

$$V_{ij} = \sum_{k=1}^{n} x_{ij(k)}^{\hat{\beta}}$$

$$V_{i.} = \left(\prod_{j=1}^{b} V_{ij} \right)^{1/b}$$

$$V_{.j} = \left(\prod_{i=1}^{a} V_{ij} \right)^{1/a}$$

$$V_{..} = \left(\prod_{j=1}^{b} \prod_{i=1}^{a} V_{ij} \right)^{1/ab}$$

Likelihood ratio tests can now be constructed to test the more restrictive of these hypotheses against less restrictive alternatives. For example, to test H_5, the hypothesis that η_{ij} is constant over i and j, against the alternative that all η_{ij} differ, one would calculate

$$\ln \lambda = \ln L(H_5) - \ln L(H_1)$$

where $L(H_k)$ denotes the likelihood function evaluated using η_{ij} and β estimated by the methods appropriate for hypothesis H_k.

A useful sequence begins by testing H_2 against H_1 to assess the hypothesis $H_0 : c_{ij} = 1$, i.e. no interaction. If H_0 is rejected, i.e., there is interaction, no other tests are performed. If there is no interaction, one would then test H_3 against H_2 and H_4 against H_2 to determine respectively whether only column or row effects are real.

An alternate sequence would be to test H_5 against H_1. In this case, if H_0 is accepted, no further tests are performed and it is concluded that neither row, column nor interaction effects are significant. If H_0 is rejected, H_5 could then be tested against H_2. If this is not significant, there is interaction and testing ceases. If it is significant, there is a row or column effect, or both. One then tests H_5 against H_3 and H_4.

In either sequence of tests, H_2 is crucial for the test of interaction. The equation for estimating the shape parameter under H_2 is characteristically different from the estimating equations under the other four hypotheses and special numerical methods will need to be developed for calculating $\hat{\beta}_2$.

Under H_1, the estimation equation is the same as for a single factor experiment in which k = ab tests are run. Similarly, under H_5 the estimating equation is identical to that which applies to a single sample of size N = abn. That is, one has only to combine the data in all ab cells of the design into a single sample and estimate the shape parameter of that single sample to obtain $\hat{\beta}_5$.

The estimating equation for $\hat{\beta}_3$ is of the same form as the single factor experiment with k = a and with the data in each row combined, i.e. ignoring the columns. Similarly for $\hat{\beta}_4$ the rows are ignored and the data within each of the columns is treated as a single group in a multiple group sample with k = b.

Thus, in the absence of interaction, the shape parameter estimates required for testing row and column effects can be obtained by arranging the data in various ways using only the software for ML estimation of the Weibull shape parameter in k groups. The appropriate values of k are 1, a, b, and ab. Moreover, with little if any loss of power, the testing can be based on just these shape parameter estimates to avoid the need for additional software to calculate the likelihood function.

We have, accordingly, generated the required tables for 2 X 2, 2 X 3, and 3 X 3 factorial arrangements with the sample sizes n and censoring number r tabled below:

Rows X Columns	n	r
2 X 2	3	3
	4	4
	5	3
	5	5
	10	5
	10	10

Rows X Columns	n	r
2 X 3	3	3
	4	2
	4	4
	5	2
	5	3
	5	5
	7	7
3 X 3	3	3
	4	4
	5	5

Under the assumption that interaction is negligible ($c_{ij} = 1$), the analysis proceeds as follows:

1. Combine all data, calculate $\hat{\beta}_5$.
2. Treat each cell as a separate sample, calculate $\hat{\beta}_1$.
3. Treat each row as a separate sample (ignore columns), calculate $\hat{\beta}_3$.
4. Treat each column as a separate sample (ignore rows), calculate $\hat{\beta}_4$.
5. Calculate $\hat{\beta}_1/\hat{\beta}_5$. If greater than its critical value, row effects or column effects or both are significant.
6. Calculate $\hat{\beta}_1/\hat{\beta}_3$. If significant, column effect is real.
7. Calculate $\hat{\beta}_1/\hat{\beta}_4$. If significant, row effect is real.

Table 2 shows the results of the analysis of a portion of a randomized block design for rolling contact endurance testing performed by Ku et al. (1). In this experiment, it was desired to determine whether there was a difference between two oils meeting the specifications MIL-L-7808 and MIL-L-23699 with respect to their influence on fatigue life in rolling contact. Ten specimens were run to failure with each lubricant on each of ten test machines. We have arbitrarily selected test machines Nos. 1 & 2 to form a 2 X 2 layout.

The values shown in each cell are (1) the ML estimate of the tenth percentile $x_{0.10}$ obtained using the ten data values taken at the conditions corresponding to that cell. (2) The ML estimate of $x_{0.10}$ under the assumption that all cells have a common shape parameter and (3) the ML shape parameter estimate based on cell data.

The values of $\hat{\beta}_1$ and $\hat{\beta}_5$ are shown in the center of the layout in Table 2. $\hat{\beta}_3$ and $\hat{\beta}_4$ are shown between the rows and columns respectively. For reference, the shape parameter estimate using the data for each tester are shown at the right hand side of each row. The shape parameter estimates using the combined data for each oil are given below each column.

To test the homogeneity of shape parameters, an assumption of the analysis, one computes the ratio of the largest to

	MIL-L-7808		MIL-L-23699	
TESTER NO. 1	$\hat{x}_{0.10} = 95.9$ $\hat{x}_{0.10} = 85.5$ $\hat{\beta} = 5.39$	$\boxed{\hat{\beta}_1 = 4.366}$	$\hat{x}_{0.10} = 102.1$ $\hat{x}_{0.10} = 161.0$ $\hat{\beta} = 5.75$	TESTER NO. 1 $\hat{\beta} = 5.54$
		$\boxed{\hat{\beta}_5 = 2.899}$		$\boxed{\hat{\beta}_3 = 4.031}$
TESTER NO. 2	$\hat{x}_{0.10} = 114.5$ $\hat{x}_{0.10} = 88.6$ $\hat{\beta} = 2.94$	$\boxed{\hat{\beta}_4 = 2.97}$	$\hat{x}_{0.10} = 164.1$ $\hat{x}_{0.10} = 132.5$ $\hat{\beta} = 6.86$	TESTER NO. 2 $\hat{\beta} = 3.48$
	MIL-L-7808 $\hat{\beta} = 2.54$		MIL-L-23699 $\hat{\beta} = 4.02$	

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TABLE 2. ANALYSIS OF ROLLING CONTACT FATIGUE DATA

smallest of the cell shape parameter estimates.

$$w = 6.86/2.94 = 2.33$$

The critical value for a 10% level test based on four samples of size n = 10 having r = 10 failures is (2).

$$w_{0.90}(n = 10,\ r = 10,\ k = 4) = 2.47$$

The hypothesis of homogeneous shape parameters is accepted but just barely.

Proceding formally we form

$$\hat{\beta}_1/\hat{\beta}_5 = 1.506$$

This is substantially greater than the 10% critical value $(\hat{\beta}_1/\hat{\beta}_5)_{0.90} = 1.085$ so that row, column or both effects are significant. Of course, an interaction effect, suggested by the common β estimates of $x_{0.10}$, also would contribute to a high value of the test statistic.

To test the difference in oils, we calculate $\hat{\beta}_1/\hat{\beta}_3 = 1.083$. The 10% level critical value is $(\hat{\beta}_1/\hat{\beta}_3)_{0.90} = 1.085$. Thus, the lubricant difference is virtually significant.

To assess the difference between testers, we compute $\hat{\beta}_1/\hat{\beta}_4 = 1.47$. This greatly exceeds the critical value 1.085, suggesting a strong tester effect. This effect is not suggested by the common β shape parameter estimates of $x_{0.10}$ These estimates in fact suggest an interaction effect.

The marginal shape parameter estimates suggest an interaction as well, for if the lube effect were the same with each tester, the shape parameter estimates within each tester would be homogeneous. The ratio of the tester shape parameter estimates is 5.54/3.48 = 1.59. This exceeds the critical value $w_{0.90}(20,20,2)$ for a 10% level test of the homogeneity of two shape parameters based on censored samples of size 20.

This effect is also ascribable to the low shape parameter estimate for the Tester No. 2 - MIL-L-7808 cell.

We conclude that, as in the analysis of variance for normal distribution theory, inhomogeneous variance (shape parameter) can cause specious results. A second application was made to rolling contact fatigue data taken at two loads and with two radii of curvature (3). These data also exhibited a nonhomogeneous shape parameter.

A paper will be prepared on the analysis of randomized block designs with Weibull response.

5.0 GROUPED DATA

Computer program WEIBSIM for simulating sets of Weibull distributed data has been modified to form a new program GROUPSIM for the analysis of grouped Weibull data.

The program generates one or more ungrouped samples, allocates each failed item to an interval, replaces its actual value by the cell midpoint value, and then performs conventional maximum likelihood estimation using the data thus modified.

The program assumes type II censoring at the r-th failure. The implication is that testing stops when a pre-established failure occurs. In actual testing, additional failures could occur prior to the end of the interval containing the r-th failure.

The intervals have been chosen logarithmically. An initial interval 'DELT' is input to the program along with a factor, 'FAC'.

The first interval extends from 0 to DELT. The second interval extends from DELT to DELT X FAC. The terminus of the I-th interval is calculated as

$$\Delta(I) = DELT \times (FAC)^{I-1}$$

or recursively as

$$\Delta(I) = FAC \times \Delta(I-1)$$

The program samples from a Weibull population having shape parameter $\beta = 1$ and a p-th percentile $x_p = 1.0$ for specified p. The simulation results apply to any 2 parameter Weibull distribution if the terminus x_i of the i-th interval satisfies

$$(x_i/x_p)^{\beta} = \Delta (I)$$

For the distribution used in the simulation, the 1st and 99-th percentiles are 0.0954 and 43.70 when using p = 0.10.

GROUPSIM was used to determine the distribution of the pivotal quantities $\hat{\beta}/\beta$ and $\hat{\beta}\ln(\hat{x}_{0.10}/x_{0.10})$ for a single uncensored sample of size n = 30, using DELT = 0.1 and FAC = 2.0, 1.5, and 1.2.

The 5-th, 50-th and 95-th percentiles are tabled below along with the corresponding ungrouped values.

	$\hat{\beta}/\beta$			$\hat{\beta}\ln(\hat{x}_{0.10}/x_{0.10})$		
	0.05	0.50	0.95	0.05	0.50	0.95
FAC = 2.0	0.900	1.056	1.269	-0.346	0.138	0.742
FAC = 1.5	0.927	1.073	1.267	-0.306	0.132	0.672
FAC = 1.2	0.951	1.080	1.239	-0.234	0.131	0.557
UNGROUPED	0.826	1.057	1.335	-0.567	0.0536	0.915

To assess the sample size effect the values for n = r = 5 and n = r = 50 with DELT = 0.1 and FAC = 1.5 are tabled below.

	$\hat{\beta}/\beta$			$\hat{\beta}\ln(\hat{x}_{0.10}/x_{0.10})$		
	0.05	0.50	0.95	0.05	0.50	0.95
n = r = 5	0.768	1.066	1.654	-0.869	-0.129	0.913
UNGROUPED	0.680	1.235	2.815	-1.142	0.447	4.445
n = r = 50	0.937	1.060	1.213	-0.252	0.121	0.564
UNGROUPED	0.852	1.018	1.235	-	-	-

The following effects are observed:

(1) There is consistently less variability in the grouped data results than the associated ungrouped values.

(2) For fixed sample size, the variability decreases with the interval width as expected, but does not appear to be converging toward the ungrouped results.

(3) The difference between grouped and ungrouped percentage decreases with sample size, i.e. convergence with sample size appears to take place.

Superficially it appears that grouping the data results in greater precision in estimating the parameters than ungrouped data, a counter intuitive result. The grouping, however, assumes perfect information regarding the parameters for the purpose of standardizing the intervals. That is, the values of $(x_i/x_{0.10})^{\beta}$ defining the interval end points is assumed known, whereas, in fact, only the x_i are known. It is recommended that this effect be examined in future studies. It is further recommended that the results given herein, obtained by conventional ML estimation using adjusted data, be compared in future studies to those obtained with direct ML estimation using a grouped data formulation.

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APPENDIX I

TECHNICAL REPORT

ANALYSIS OF TIME-TO-EVENT DATA

SUPPLIED BY EGLIN AFB

by

John I. McCool

SKF Report No. AL79P026

PREPARED UNDER:

AFOSR-F49620-79-C-0035
BOLLING AFB, D.C.20332

SKF INDUSTRIES, INC.
TECHNOLOGY SERVICES DIVISION
1100 FIRST AVENUE
KING OF PRUSSIA, PA 19406

TECHNICAL REPORT

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ANALYSIS OF TIME-TO-EVENT DATA SUPPLIED BY EGLIN AFB

1.0 Introduction and Summary

A methodology has recently been developed under the sponsorship of the Air Force Office of Scientific Research for the unbiased point and interval estimation of the location parameter of a three-parameter Weibull distribution.

It has been suggested [1] that time-to-event data of the type encountered in fuzing mechanisms may follow a three-parameter Weibull distribution. If this is so the location or threshold parameter of the distribution represents the "safe" time prior to which the ignition event can not occur.

In this context a lower confidence limit on the location parameter represents a quantifiably conservative estimate of the "safe time" for the device.

This report describes the analysis of a sample of 29 unidentified time-to-event observations supplied by Eglin Air Force Base.

Under the assumption that the data are drawn from a three-parameter Weibull population, a median unbiased estimate and a lower 95% confidence limit for the location parameter have been calculated using computer program "LOCEST" implementing the methodology referred to above. Subtracting the median unbiased location parameter estimate from each event time and regarding the data thus adjusted as a two-parameter Weibull sample, computer program "WEIBEST" was used to estimate the shape and scale parameter by the method of maximum likelihood. A chi-square goodness of fit test was then performed and indicated that the fitted three-parameter Weibull distribution did not adequately describe the data. Instead, the data appear to consist of a mixture of two Weibull distributions with one population, representing roughly 86% of the data, having a shape parameter of $\beta = 18.6$ and a second population representing 14% of the observations and having a shape parameter estimated graphically to be $\beta = 1.4$.

The data thus support the assumption that the population from which the sample was drawn consist of a mixture of effective items having a two-parameter Weibull distribution and a 10-20% subpopulation of "duds" for which the event occurs at random intervals in accordance with a poisson process. To estimate the "safe" time associated with some arbitrary low event probability, the long lived items should be censored prior to estimating the Weibull parameters. If this is not done the Weibull shape parameter for the effective items will be underestimated and overly conservative safe lives will be computed.

Section 2.0 of this report describes the methodology for location parameter estimation. Section 3.0 gives the details of the analysis of the Eglin time-to-event data.

2.0 Methodology

2.1 The Weibull Distribution

The three parameter Weibull distribution has the cumulative form:

$$Prob[X < x] = F(x) = 1 - exp\,[-(x-\gamma)/\eta)^{\beta}] \qquad ;\ x > \gamma \tag{1}$$

where γ = location parameter

η = scale parameter

β = shape parameter

The two-parameter Weibull distribution is the special case of Equation (1) in which the location parameter $\gamma = 0$.

2.2 Graph of Two and Three Parameter Weibull Functions

For the two-parameter Weibull distribution it is readily shown that

$$y(x) \equiv \ell n \ell n(1/(1-F(x)) = \beta\ \ell n\ x - \beta\,\ell n\ \eta \tag{2}$$

Thus, in the two-parameter case $y(x)$ is a linear function of $\ell n\ x$ having slope β and intercept $-\beta\ell n\eta$. For the three-parameter case

$$y(x) \equiv \ell n\ \ell n\ (1/(1-F(x)) = \beta\ \ell n(x-\gamma) - \beta\ \ell n\ \eta \qquad ;\ x \geq \gamma \tag{3}$$

The slope of a plot of $y(x)$ against ℓnx in the three-parameter case is

$$\frac{dy(x)}{d\ell n(x)} = \frac{\beta x}{x-\gamma} \qquad ;\ x \geq \gamma \tag{4}$$

The slope is infinite at $x = \gamma$ and decreases monotonically thereafter to an asymptote of β. Figure 1 is a sketch of $y(x)$ plotted against $\ell n x$ for $\gamma = 0$ and $\gamma > 0$.

2.3 Graphical Estimation of Shape Parameter

Let $x_1 < x_2 < x_3 < \ldots x_n$ denote the ordered observations in a random sample of size n drawn from a two or three-parameter Weibull distribution. An estimate of $f(x_i)$ may be calculated for each of the ordered observations using any of the various choices of plotting position. A common choice is:

$$\hat{F}(x_i) = i/(n + 1) \tag{5}$$

An estimate $\hat{y}(x_i)$ may then be computed by substituting $\hat{F}(x_i)$ into Equation (2).

If the sample is drawn from a two-parameter Weibull distribution $\hat{y}(x_i)$ will tend to plot against $\ell n x_i$ as a straight line with slope β. If $\gamma > 0$, i.e. the population is a three-parameter Weibull distribution, $\hat{y}(x_i)$ will tend to be a concave function of $\ell n x_i$ approaching a constant slope β for large x_i values.

Figure 2 shows how a plot of $\hat{y}(x_i)$ vs. $\ell n x_i$ might appear for a sample drawn from a three-parameter Weibull distribution.

If these data were regarded as a two-parameter Weibull sample a graphical estimate of the shape parameter $\hat{\beta}_A$ could be found as the slope of the straight line that best fits the complete data sample.

If only a subset of the smallest ordered values were used in graphically estimating the shape parameter, the estimate $\hat{\beta}_L$ would be obtained. For three parameter Weibull data $\hat{\beta}_L$ will

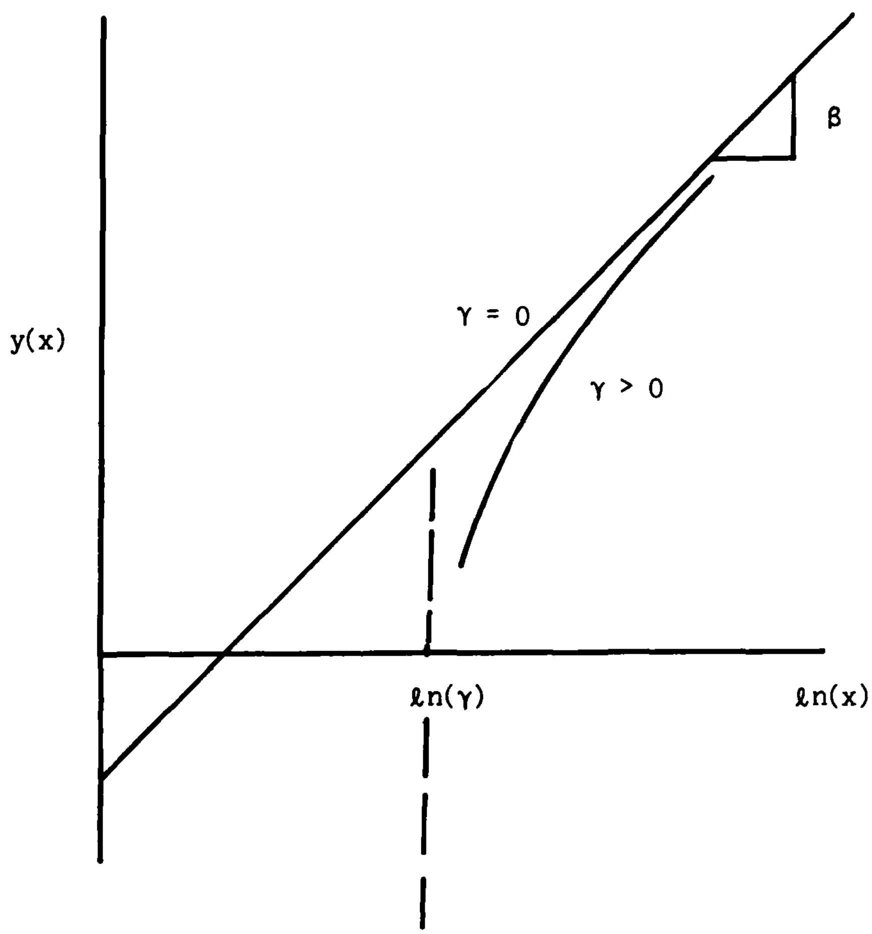

FIGURE 1. PLOT OF y(x) AGAINST ℓn(x) FOR $\gamma = 0$ AND $\gamma > 0$

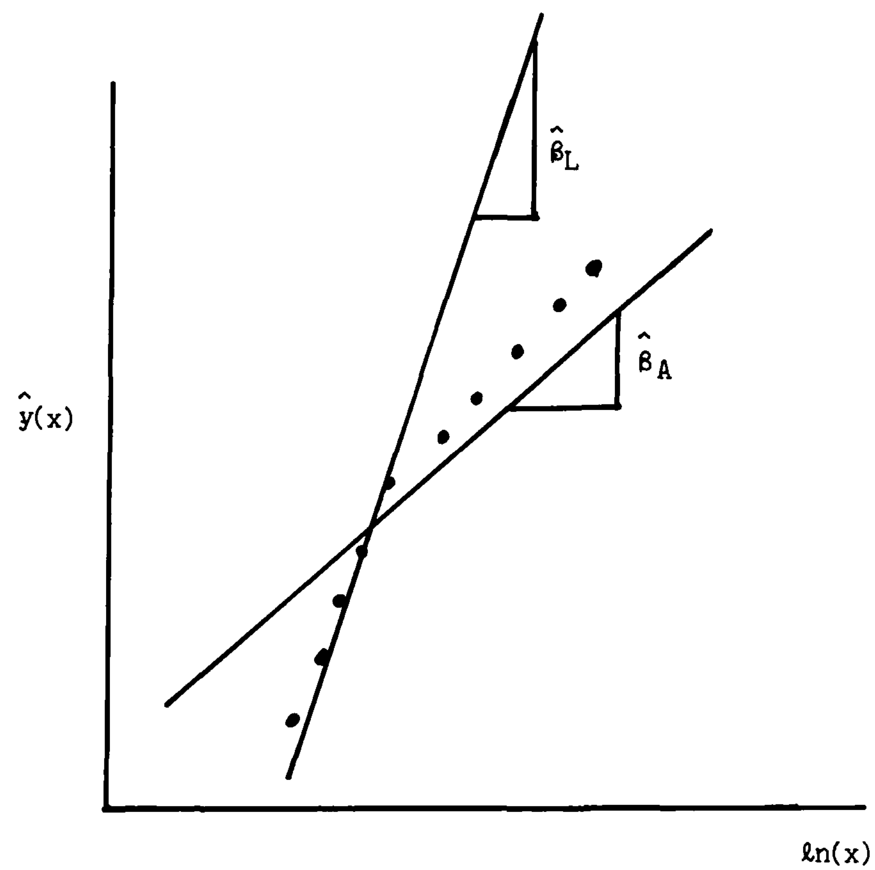

FIGURE 2. GRAPHICAL ESTIMATION OF WEIBULL SHAPE PARAMETER

tend to exceed $\hat{\beta}_A$. On the other hand when the sample is drawn from a two-parameter Weibull population $(\gamma=0)$, $\hat{\beta}_L$ and $\hat{\beta}_A$ will be comparable.

2.4 Maximum Likelihood Estimation of Shape Parameter

Rather than graphical estimation we consider maximum likelihood (ML) estimation of β.

For a sample of size n censored at the r-th ordered observation x_r, the ML estimate of β for a two-parameter Weibull distribution is the solution of the nonlinear equation:

$$1/\hat{\beta} + \sum_{i=1}^{r} \log x_i/r - \left(\sum_{i=1}^{r} x_i^{\hat{\beta}} \log x_i + x_r^{\hat{\beta}}(n-r)x_r\right)/\left(\sum_{i=1}^{r} x_i^{\hat{\beta}} + (n-r)x_r^{\hat{\beta}}\right) = 0 \qquad (6)$$

It has been shown (cf. McCool [2]) that $\hat{\beta}/\beta$ is a pivotal function, i.e. it follows a distribution that depends on n and r but not on the underlying Weibull population parameters.

Denoting the solution of Eq. (6) as $\hat{\beta}_A$ and the solution of Eq. (6) with $r_1<r$ as $\hat{\beta}_L$, the distribution of $w = \hat{\beta}_A/\hat{\beta}_L$ will depend only upon n, r_1, and r when the underlying distribution is indeed of the two-parameter Weibull form. When the underlying distribution is the three-parameter Weibull the mean value of $\hat{\beta}_L$ will increase proportionally more than the mean value of $\hat{\beta}_A$.

With the percentiles of w determined by Monte Carlo sampling for specified r_1, r and n, one may reject the hypothesis that $\gamma = 0$ at the 100α% level if

$$\hat{\beta}_A/\hat{\beta}_L > w_{1-\alpha} \qquad (7)$$

2.5 Interval Estimation

Given that a random variable x is drawn from a three-parameter Weibull population having location parameter γ the transformed variable $y = x-\gamma$ will follow a two-parameter Weibull distribution with the same scale and shape parameters as the three-parameter distribution. Thus, if γ is subtracted from the observed data prior to calculating $\hat{\beta}_A$ and $\hat{\beta}_L$ from Eq. (6), the resulting ratio, denoted

$$w(\gamma) = \hat{\beta}_A/\hat{\beta}_L \tag{8}$$

will follow the null distribution of w determined by Monte Carlo sampling from a two-parameter Weibull population for given values of n, r_1 and r.

We may thus write the 100(1-α)% probability statement

$$Prob\ [w(\gamma) < w_{1-\alpha}] = 1 - \alpha \tag{9}$$

We also need the fact, heuristically suggested by the analogy to graphical estimation, that if an amount λ is subtracted from each observation in a given sample prior to calculating $\hat{\beta}_A$ and $\hat{\beta}_L$, $w(\lambda) = \hat{\beta}_A/\hat{\beta}_L$ will be a decreasing function of λ. Accordingly, we may invert the inequality of Eq. (9) to give a 100(1-α)% lower confidence limit for γ, i.e.

$$\gamma > w^{-1}\ \{w_{1-\alpha}\} \tag{10}$$

3.0 Analysis of Eglin Data

3.1 Analysis as a Two-Parameter Weibull Sample

Table 1 is the output of computer program WEIBEST which was applied to the raw Eglin data. The tabular output at the top of Table 1 is a sorted list of the 29 observations of time-to-event in seconds.

The first line below the sorted times gives the maximum likelihood estimates of the tenth and 50-th percentiles (designated L_{10} and L_{50}, respectively) and the Weibull shape parameter β, computed under the assumption that the data were drawn from a two-parameter Weibull distribution. Subsequent lines in Table 1 give lower and upper 90% confidence limits and median unbiased estimates of L_{10}, L_{50} and β.

Figure 3 shows a probability plot of the data using scales on which two parameter Weibull samples tend to plot as a straight line. The fitted two-parameter population is shown as a solid straight line and is clearly a poor fit to the data. The two dotted straight line segments fitted to the data are discussed further below.

3.2 Analysis as a Three-Parameter Weibull Sample

Figure 4 is a plot of the function $w(\lambda)$ computed from Eq. (8) for positive λ values. In calculating this plot r_1 was taken as 5 and r_2 as 29. The plot decreases with λ, approaching a vertical asymptote as λ approaches the first order statistic $x_{(1)}$.

TABLE 1

WEIBEST OUTPUT RAW DATA

TIME-TO-EVENT DATA FROM EGLIN AFB

Group No. 1 Lives

5.2200	6.0700	6.5500
5.4100	6.1500	6.5600
5.7300	6.2600	6.5900
5.7800	6.3100	6.6000
5.7900	6.3400	6.7300
5.8700	6.3500	7.1300
5.9200	6.4300	7.8300
5.9800	6.4400	9.2800
6.0000	6.4500	10.2800
6.0300	6.4600	

L_{10}	L_{50}	BETA
0.4556E 01	0.6501E 01	0.5299E 01
LCL L_{10}	MED L_{10}	UCL L_{10}
0.3748E 01	0.4524E 01	0.5113E 01
LCL L_{50}	MED L_{50}	UCL L_{50}
0.6024E 01	0.6499E 01	0.6912E 01
LCL BETA	MED BETA	UCL BETA
0.3893E 01	0.5125E 01	0.6592E 01

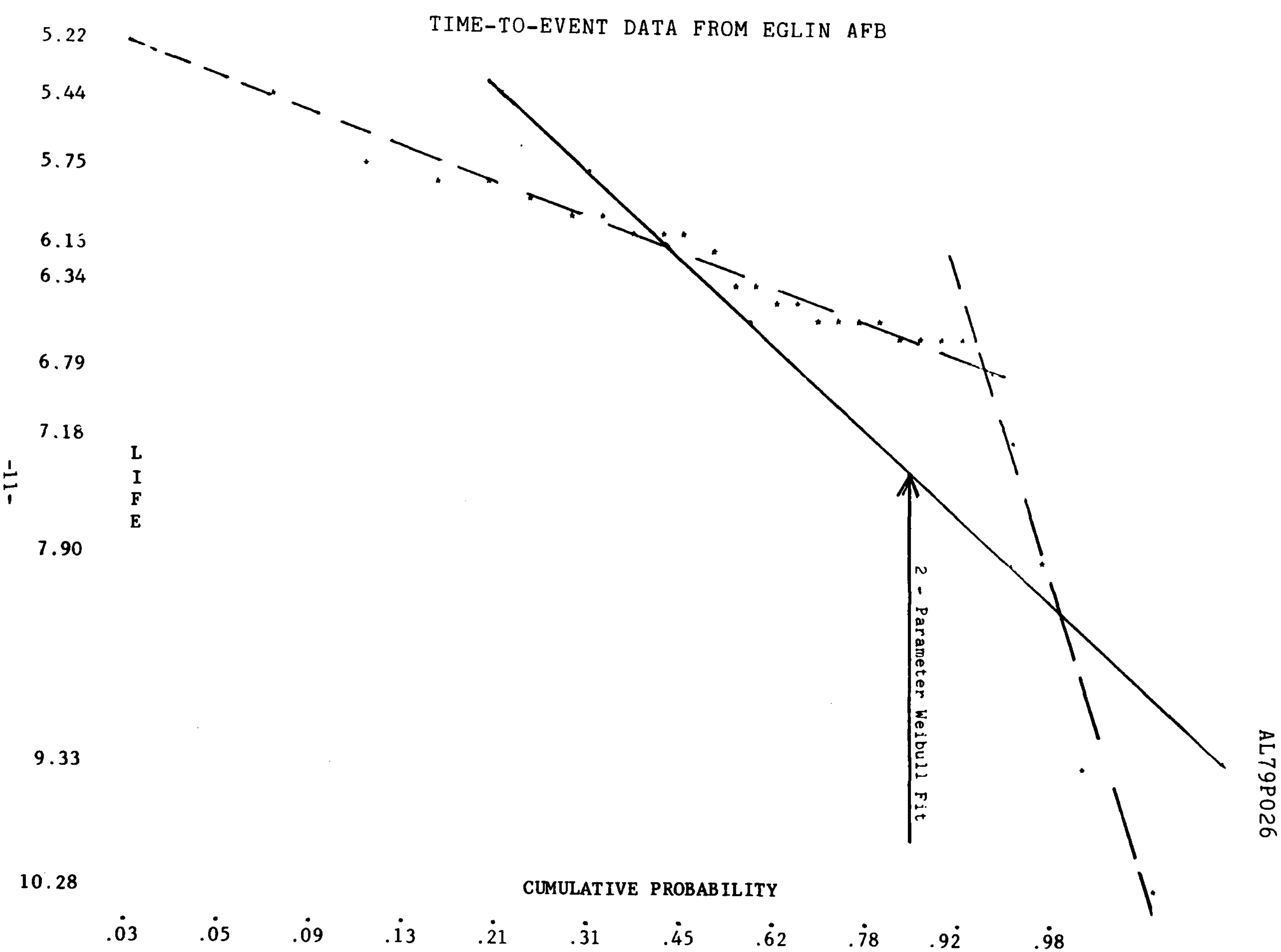

FIGURE 3. WEIBULL PROBABILITY PLOT OF RAW DATA

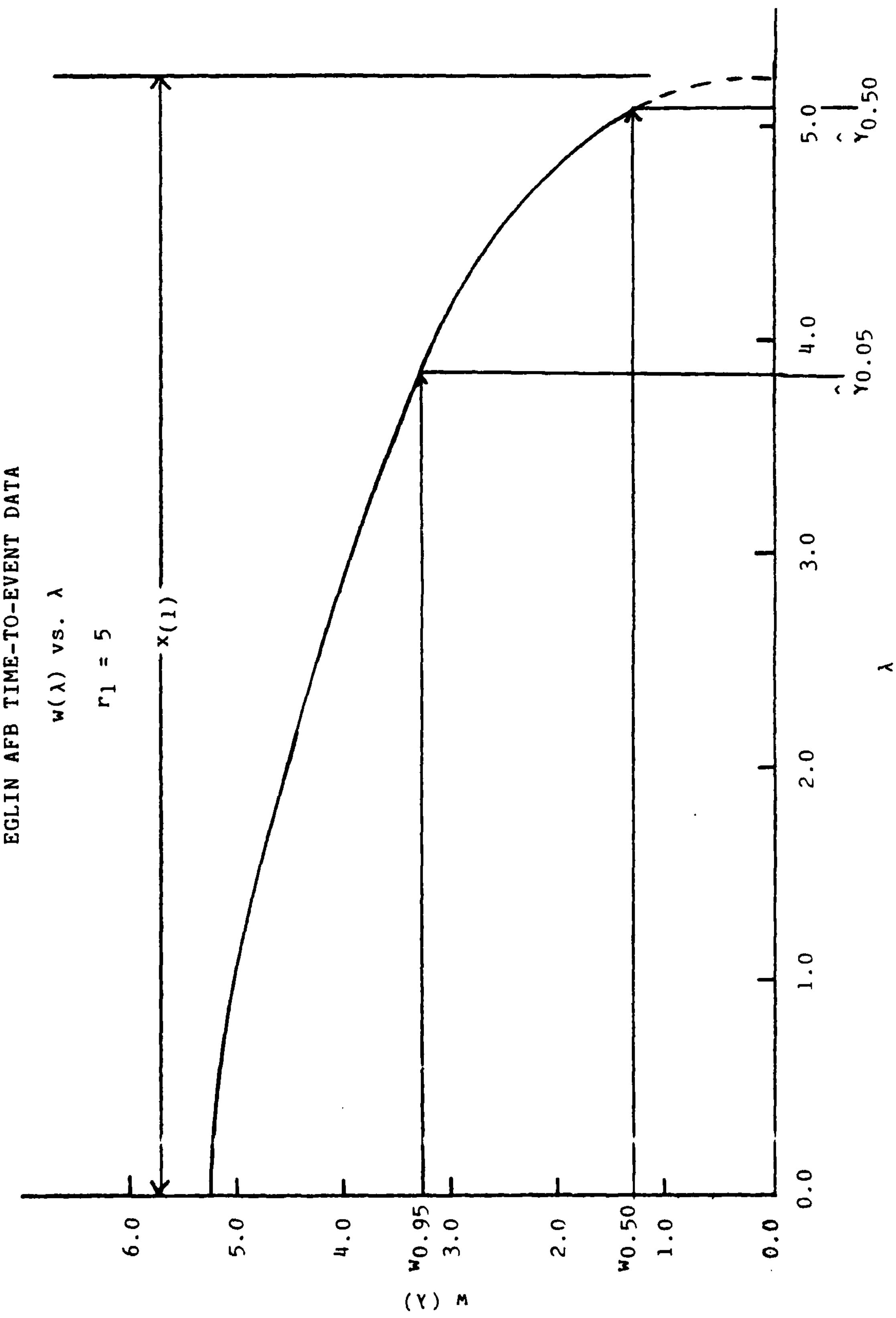

FIGURE 4. THE FUNCTION w (λ)

EGLIN AFB TIME-TO-EVENT DATA

w(λ) vs. λ

$r_1 = 5$

The median unbiased estimate $\hat{\gamma}_{0.50}$ is shown to be the λ value associated with $w(\lambda) = w_{.50}$ and the 95% lower confidence limit $\hat{\gamma}_{0.05}$ is the λ value corresponding to $w_{0.95}$.

Note that if $w(0) < w_{0.95}$, a positive value of $\hat{\gamma}_{0.05}$ cannot be found.

Computer program LOCEST calculates $\hat{\gamma}_{0.50}$ and $\hat{\gamma}_{0.05}$ for specified values of $w_{.95}$ and $w_{.50}$ using a golden section search technique. The appropriate values found from Monte Carlo simulation for $n=30$, $r_1=5$, $r=30$ are

$$w_{0.50} = 1.294$$

$$w_{.95} = 3.279$$

The values for $n=29$, $r=29$ are not likely to differ substantially from these. Table 2 shows the LOCEST output.

The Weibull shape parameter considering 24 items censored at the 5-th smallest time is

$$\hat{\beta}_L = 27.9$$

Using all the data the estimate is

$$\hat{\beta}_A = 5.30$$

The ratio $\hat{\beta}_L/\hat{\beta}_A = 5.27$ corresponds to $w(o)$ and since $w(o)$ exceeds $w_{0.50}$ and $w_{0.95}$, positive values of both $\hat{\gamma}_{0.50}$ and $\hat{\gamma}_{0.05}$ may be calculated.

These values are $\gamma_{0.05} = 3.88$

and $\gamma_{0.50} = 5.09$

The two parameter estimates of $x_{0.10}=L_{10}-\hat{\gamma}_{.50}$ and $x_{0.50} =L_{50}-\gamma_{.50}$ are 0.364 and 1.243, respectively. The estimated shape parameter from the adjusted data is 1.53.

TABLE 2

LOCEST OUTPUT

TIME-TO-EVENT DATA FROM EGLIN AFB

WEIBULL LOCATION PARAMETER ESTIMATION

SAMPLE SIZE,N= 29

TRUNCATION NUMBER FOR LOCATION PARAMETER ESTIMATION,R1= 5

NUMBER OF FAILURES,R= 29

W50= 1.294

W95= 3.279

BETA HAT(R1)= 27.940

BETA HAT(R) = 5.299

BETA HAT(R1)/BETA HAT(R)= 5.272

MEDIAN UNBIASED ESTIMATE OF GAMMA= 5.091

LOWER 95% CONFIDENCE LIMIT FOR GAMMA= 3.881

ADJUSTED MAXLIKE ESTIMATES OF 10-TH AND 50-TH PERCENTILE

X0.10= 0.364

X0.50= 1.243 ADJUSTED BETA = 1.533

Figure 5 is a probability plot of the data after adjustment by subtraction of $\hat{\gamma}_{0.50}$, with the fitted distribution shown as a solid straight line. A chi-square goodness of fit test was applied and led to rejection of the hypothesis that the fitted distribution actually represented the data (cf. Appendix). These dashed line segments emphasize that the behavior in the two tails is inconsistent with the fitted distribution.

Returning to Figure 3, we note that the two dashed line segments together fit the observed data very well and suggest that the data may be a mixture of two Weibull populations; one population having a high shape parameter value and low mean time-to-event and a second population having a much lower shape parameter value and a high mean time-to-event. For this data sample 86% of the sample belongs to the first population.

To estimate the parameters of the first population, the data were censored at the 24-th event time and WEIBEST was rerun. The results are given in Table 3.

The shape parameter is estimated to be 18.6. This is much higher than the value 5.3 shown in Table 2 based on all the data.

Figure 6 is a probability plot of the censored data and confirms the good fit of the two-parameter Weibull population to the bulk of the early events.

A graphical estimate of the shape parameter for the long event time items is 1.4. This is consistent with a shape parameter of 1.0 which suggests that the long event time population may have an exponentially distributed time between failures characteristic of a poisson process governing the occurrence of events. This suggests that the late events correspond to a sub-

TABLE 3

WEIBEST OUTPUT FOR CENSORED DATA

EGLIN DATA CENSORED AT 24-TH ORDERED OBSERVATION

Group No. 1 Lives

5.2200	6.0700	6.5500
5.4100	6.1500	6.5600
5.7300	6.2600	6.5900
5.7800	6.3100	6.6000S
5.7900	6.3400	6.6000S
5.8700	6.3500	6.6000S
5.9200	6.4300	6.6000
5.9800	6.4400	6.6000S
6.0000	6.4500	6.6000S
6.0300	6.4600	

L_{10}	L_{50}	BETA
0.5695E 01	0.6301E 01	0.1863E 02
LCL L_{10}	MED L_{10}	UCL L_{10}
0.5366E 01	0.5677E 01	0.5870E 01
LCL L_{50}	MED L_{50}	UCL L_{50}
0.6166E 01	0.6303E 01	0.6436E 01
LCL BETA	MED BETA	UCL BETA
0.1296E 02	0.1782E 02	0.2377E 02

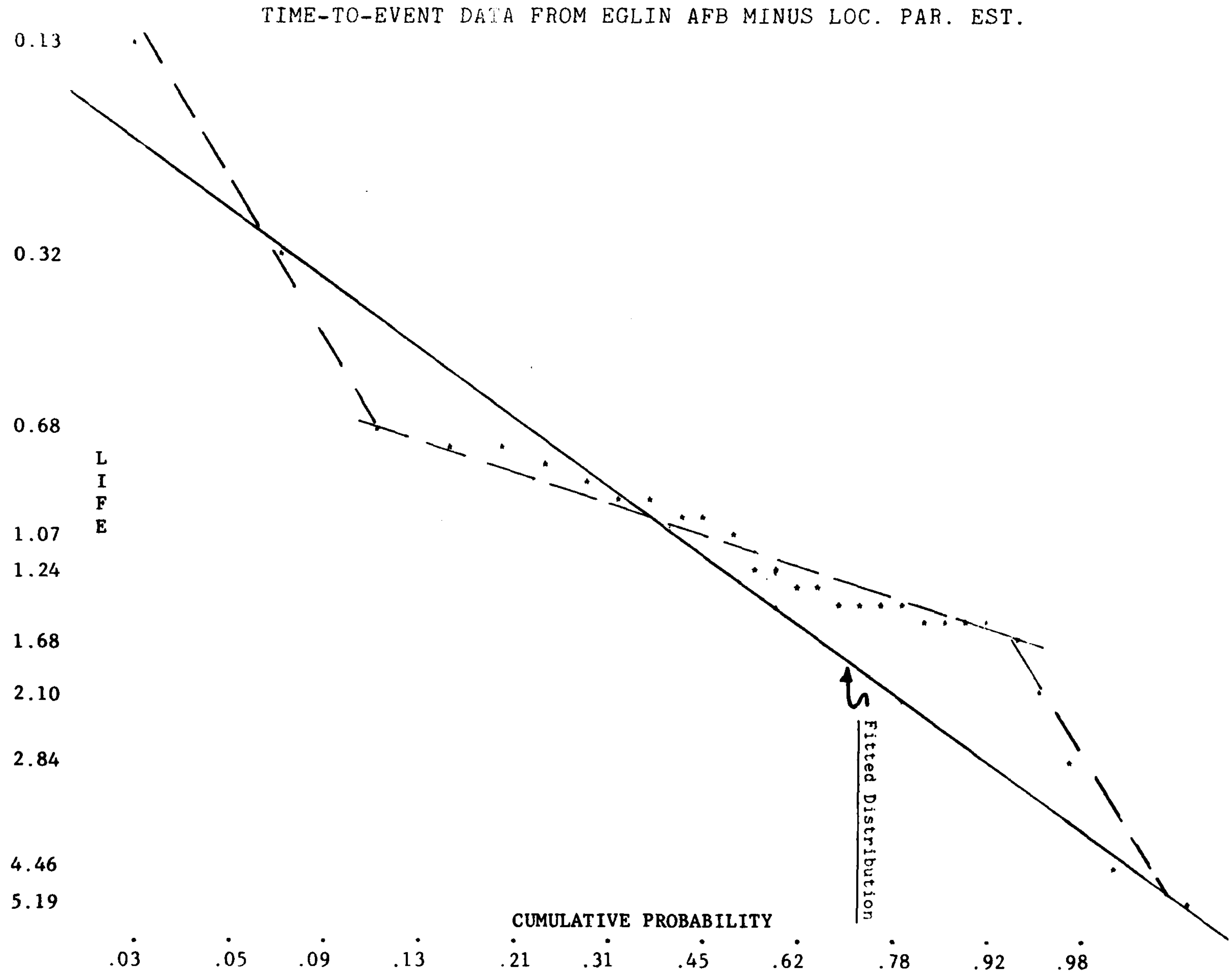

FIGURE 5. WEIBULL PROBABILITY PLOT OF ADJUSTED DATA

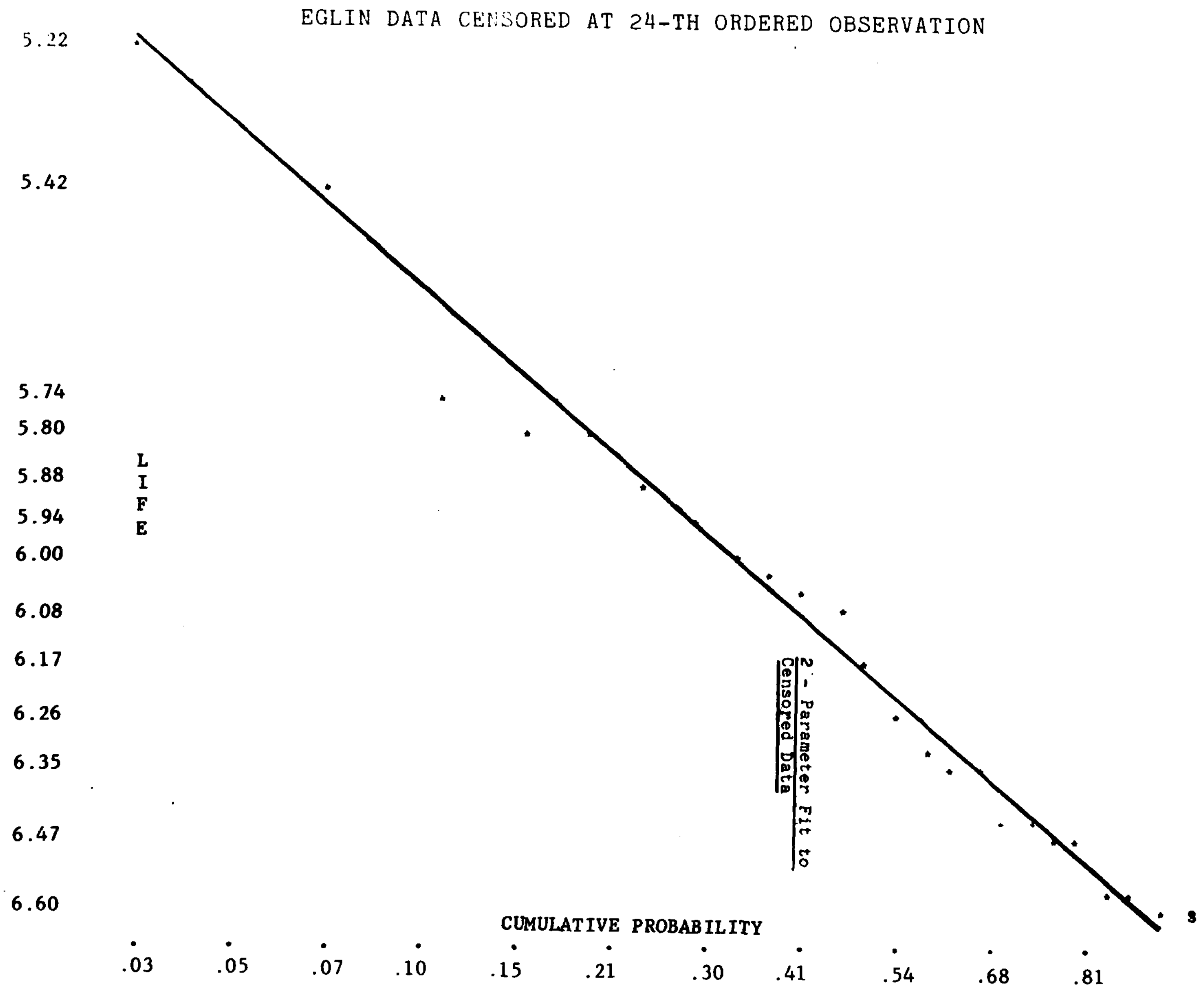

FIGURE 6. PLOT OF CENSORED TIME-TO-EVENT DATA

AL79P026

population of "duds" wherein the event is triggered by some sort of random shock rather than by the design mechanism.

3.3 Discussion of Data Analysis

The data do not confirm a three-parameter Weibull model implying a "safe" time prior to which the "event" cannot occur. A "safe" time must therefore be defined as the time associated with some arbitrary low probability that the event will occur prior to it. Because the data suggest a mixture of Weibull models the direct use of a two-parameter Weibull model, as shown in Figure 3, will result in overly conservative safe time estimates. This may be overcome by censoring 15-20% of the long time events which correspond to a population of "duds" that are mixed with the effective items.

References

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APPENDIX

GOODNESS OF FIT TEST FOR THREE-PARAMETER WEIBULL

From Tables 2 and 3 the fitted three-parameter Weibull CDF has the equation

$$F(x) = 1 - exp - 0.10536 \left[\frac{t}{0.3643}\right]^{1.534} \qquad (A.1)$$

where

$$t = x - 5.09 \qquad (A.2)$$

We divide the t axis into 5 intervals each having a 20% occurrence probability by calculating the percentiles $t_{0.20}$, $t_{0.40}$, $t_{0.60}$ and $t_{.80}$ where t_p satisfies

$$p = 1 - exp - .10536 \, [t_p/.3643]^{1.534} \qquad (A.3)$$

The expected number of observations in the i-th interval is

$$e_i = n \times p = 29 \times .20 = 5.8 \qquad (i = 1, \ldots 5) \qquad (A.4)$$

The following Table shows for each interval the observed number of observations in the interval, o_i, the expected number, e_i, and the square of the differences $(o_i - e_i)^2$.

Interval on t	o_i	e_i	$(o_i-e_i)^2$
0 - 0.5941	2	5.8	14.44
> 0.5941 - 1.020	9	5.8	10.24
> 1.020 - 1.492	11	5.8	27.04
> 1.492 - 2.154	4	5.8	3.24
> 2.154 - ∞	3	5.8	7.84

Under the hypothesis that the data were drawn from the fitted distribution, the quantity

$$u = \sum_{i=1}^{5} (o_i - e_i)^2/e_i = 56.2/5.8 = 9.69 \tag{A.5}$$

asymptotically follows a χ^2 distribution with $m-p-1$ degrees of freedom where:

m = no. of intervals

p = no. of parameters estimated by the method of maximum likelihood

This asymptotic result is generally believed to be applicable if $e_i = 5$. In the present case $m=5$ and $p=3$ so that u will be approximately distributed as $\chi^2(1)$ under the null hypothesis. From tables of the χ^2 distribution we find

$$\chi^2_{0.95}(1) = 3.84$$

Since $u = 9.69 > 3.84$ the null hypothesis is rejected at the 5% significance level.

APPENDIX II

AT80D041

INFERENCE ON THE WEIBULL LOCATION PARAMETER

J. I. McCool
SKF Industries, Inc.
King of Prussia, Pa. 19406

1. HYPOTHESIS TEST: 2 vs. 3 PARAMETER WEIBULL

The two-parameter Weibull distribution has CDF

$$F(x) = 1-\exp-[x/\eta]^{\beta}$$

The maximum likelihood estimate of the shape parameter β calculated from the ordered observations x_i in a random sample of size n type II censored at the r-th failure is denoted $\hat{\beta}(r)$ and is the solution of:

$$1/\hat{\beta} + \sum_{i=1}^{r} \ln(x_i)/r - \sum_{i=1}^{n} x_i^{\hat{\beta}} \ln(x_i) / \sum_{i=1}^{n} x_i^{\hat{\beta}} = 0$$

where $x_i = x_r$, $i > r$.

Define the random variable

$$w \equiv \hat{\beta}(r_1)/\hat{\beta}(r)$$

where $r_1 < r$. The distribution of w depends only on r_1, r and n.

When sampling from the three-parameter Weibull distribution with location parameter $\gamma > 0$, w becomes stochastically larger. The acceptance region for a $100\alpha\%$ level test of H_0: $\gamma = 0$ against against H_1: $\gamma > 0$ is:

$\hat{\beta}(r_1)/\hat{\beta}(r) < w_{1-\alpha}$

Percentage points found by Monte Carlo sampling are given in Table 1. In studies with $\beta=1$ the choice $r_1=5$ was found to be nearly optimum for all n and r in the range represented in Table 1.

2. INTERVAL ESTIMATION

Subtract $\gamma(>0)$ from each ordered observation in a random sample from a three-parameter Weibull distribution. Define

$$w(\lambda) = \hat{\beta}(r_1)/\hat{\beta}(r).$$

$w(\lambda)$ is a decreasing function of λ and $w(\lambda=\gamma)$ follows the same distribution as w calculated from two-parameter Weibull samples.

Inverting the statement: $\text{Prob}[w(\gamma) < w_{1-\alpha}]=1-\alpha$ gives the lower $100(1-\alpha)\%$ confidence limit:

$$\gamma > w^{-1}(w_{1-\alpha}).$$

3. EXAMPLE

The times to an ignition event for 29 fuzing devices are as follows:
5.22, 5.41, 5.73, 5.78, 5.79, 5.87, 5.92, 5.98, 6.00, 6.03, 6.07, 6.15, 6.26, 6.31, 6.34, 6.35, 6.43, 6.44, 6.45, 6.46, 6.55, 6.56, 6.59, 6.60, 6.73, 7.13, 7.83, 9.28, 10.28

Using $r_1=5$ and $r=29$ gives

$$\hat{\beta}(r_1) = 27.9 \text{ and } \hat{\beta}(r) = 5.30$$

n	r_1	r	$w_{0.50}$	$w_{0.90}$	$w_{0.95}$
10	2	5	1.962	12.14	23.83
10	2	6	2.129	13.36	26.96
10	2	7	2.220	14.11	28.77
10	2	8	2.332	14.80	30.50
10	2	10	2.466	16.14	32.28
10	4	10	1.251	2.757	3.728
10	5	6	0.988	1.488	1.789
10	5	7	1.035	1.730	2.132
10	5	8	1.077	1.902	2.352
10	5	9	1.116	2.022	2.517
10	5	10	1.138	2.126	2.683
10	6	10	1.075	1.772	2.150
10	7	10	1.029	1.550	1.810
15	2	5	1.956	11.75	24.47
15	2	10	2.437	15.65	32.15
15	2	11	2.463	16.05	32.74
15	2	12	2.498	16.47	33.44
15	2	15	2.603	17.12	35.04
20	5	10	1.137	2.073	2.582
20	5	12	1.172	2.198	2.784
20	5	15	1.210	2.345	2.974
20	5	18	1.238	2.417	3.135
20	5	20	1.254	2.491	3.188
20	6	10	1.070	1.726	2.052
20	6	15	1.139	1.991	2.396
20	7	10	1.030	1.518	1.767
20	7	15	1.102	1.783	2.095
20	8	10	1.004	1.349	1.517
20	9	10	0.983	1.205	1.308
20	9	15	1.045	1.500	1.702
20	9	20	1.082	1.636	1.867
20	10	15	1.026	1.405	1.568
20	10	20	1.058	1.546	1.750
20	15	20	1.006	1.261	1.374
25	5	10	1.146	2.118	2.622
25	5	14	1.199	2.321	2.924
25	5	18	1.237	2.460	3.087
25	5	20	1.250	2.515	3.148
25	5	25	1.278	2.540	3.278
30	2	20	2.639	17.11	35.87
30	2	30	2.700	17.60	36.45
30	5	10	1.139	2.079	2.602
30	5	15	1.213	2.340	2.915
30	5	20	1.256	2.457	3.119
30	5	25	1.278	2.544	3.224
30	5	30	1.294	2.600	3.279
30	7	20	1.136	1.889	2.238
30	7	30	1.170	2.024	2.405
30	8	20	1.102	1.746	2.018
30	9	30	1.112	1.756	2.012
30	10	20	1.061	1.529	1.734
30	10	30	1.092	1.657	1.900
40	2	20	2.582	17.00	34.64
40	5	20	1.245	2.496	3.084
40	7	20	1.136	1.888	2.240
40	8	20	1.099	1.731	2.017
40	10	20	1.059	1.520	1.717

TABLE 1. PERCENTAGE POINTS OF $w(\gamma)$ FOR VARIOUS n, r_1 and r

$$\hat{\beta}(r_1)/\hat{\beta}(r) = 5.27 > 3.279 = w_{0.95}(r_1=5\ ,r=30)$$

The two-parameter Weibull hypothesis is rejected in favor of the three-parameter alternative.

The median unbiased estimate of the location parameter, $\hat{\gamma}_{0.50}$ is found by solving

$$w(\hat{\gamma}_{0.50}) = w_{0.50} = 1.294$$

to be

$$\hat{\gamma}_{0.50} = 5.09$$

A lower 95% confidence limit $\hat{\gamma}_{0.50}$ was found by solving

$$w(\hat{\gamma}_{0.05}) = w_{0.95} = 3.279$$

to be

$$\hat{\gamma}_{0.05} = 3.88$$

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APPENDIX III

INFERENCE IN WEIBULL REGRESSION

J. I. McCOOL

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SKF TECHNOLOGY SERVICES
SKF INDUSTRIES, INC.
1100 FIRST AVENUE
KING OF PRUSSIA, PA 19406

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INFERENCE IN WEIBULL REGRESSION

John I. McCool, SKF Industries, Inc.
King of Prussia, PA 19406

1. THE MODEL

A random variable x is presumed to follow a two-parameter Weibull distribution with shape parameter β and scale parameter η that varies as an inverse power of a positive deterministic external variable S, generically termed a stress. That is,

$$F(x|S) = 1 - \exp - [x/\eta(S)]^{\beta} \tag{1}$$

$$\eta(S) = \eta_o \, S^{-\gamma} \tag{2}$$

with $\eta_o, \beta, \gamma, x > 0$

As a consequence of (1) and (2) the p-th quantile at stress S is

$$x_p(S) = (-\ell n\ (1-p))^{1/\beta} \cdot \eta(S) \tag{3}$$

2. MAXIMUM LIKELIHOOD (ML) ESTIMATION

A life test is carried out at each of k stress levels denoted $S_1, S_2 \ldots S_k$. At the i-th stress level n_i items are tested until the first r_i ordered failures are observed (type II testing). The ML estimates of γ and β are calculated as the simultaneous solution of the following equations.

$$\sum_{i=1}^{k} r_i \log S_i - \frac{\sum_{i=1}^{k} r_i \sum_{i=1}^{k} S_i^{\hat{\gamma}\hat{\beta}} \log S_i \sum_{j=1}^{n_i} [x_{i(j)}]^{\hat{\beta}}}{\sum_{i=1}^{k} S_i^{\hat{\gamma}\hat{\beta}} \sum^{n_i} [x_{i(j)}]^{\hat{\beta}}} = 0 \tag{4}$$

$$\frac{1}{\hat{\beta}} + \frac{\sum_{i=1}^{k}\sum_{j=1}^{r_i} \log x_{i(j)}}{\sum_{i=1}^{k} r_i} - \frac{\sum_{i=1}^{k} S_i^{\hat{\gamma}\hat{\beta}} \sum_{j=1}^{n_i} x_{i(j)}^{\hat{\beta}} \log x_{i(j)}}{\sum_{i=1}^{k} S_i^{\hat{\gamma}\hat{\beta}} \sum_{j=1}^{r_i} x_{i(j)}^{\hat{\beta}}} = 0 \qquad (5)$$

where $x_{i(j)}$ is the j-th ordered life in the i-th sample.
The ML estimate of η_o is

$$\hat{\eta}_o = \{ \sum_{i=1}^{k} \sum_{j=1}^{n_i} (\frac{x_{i(j)}}{S_i^{-\hat{\gamma}}})^{\hat{\beta}} / \sum_{i=1}^{k} r_i \}^{1/\hat{\beta}} \qquad (6)$$

The ML estimates of $\eta(S)$ or $x_p(S)$ are given by Eqs. (1) and (2) on substituting the ML estimates of γ, β and η_o.

3. PIVOTAL FUNCTIONS

The following functions are "pivotal", i.e. they follow distributions that do not depend on the population values of the parameters:

$$q = \hat{\beta}/\beta \qquad (7)$$

$$w^* = (\hat{\gamma} - \gamma)\hat{\beta} \qquad (8)$$

$$u^* = \hat{\beta}\ln [\hat{x}_p(S)/x_p(S)] \qquad (9)$$

The distribution of q and w* depends on k, S_i, n_i and r_i. The distribution of u* depends additionally upon p and S. For fixed choices of these parameters the distributions may be determined by Monte Carlo sampling.

4. INTERVAL AND MEDIAN UNBIASED POINT ESTIMATES

Two sided 100 (1-α)% interval estimates of β, γ and $x_p(S)$ may be calculated in terms of the ML estimates and the percentage points of q, w* and u* as:

$$\hat{\beta}/q_{(1-\alpha/2)} < \beta < \hat{\beta}/q_{\alpha/2} \tag{10}$$

$$\hat{\gamma} - w^*_{1-\alpha/2}/\hat{\beta} < \gamma < \hat{\gamma} - w^*_{\alpha/2}/\hat{\beta} \tag{11}$$

$$\hat{x}_p(S) \cdot \exp\,[-u^*_{1-\alpha/2}/\hat{\beta}] < x_p(S) < \hat{x}_p(S) \cdot \exp\,[-u^*_{\alpha/2}/\hat{\beta}] \tag{12}$$

Median unbiased point estimates are calculable as:

$$\hat{\beta}' = \hat{\beta}/q_{0.50} \tag{13}$$

$$\hat{\gamma}' = \hat{\gamma} - w^*_{0.50}/\hat{\beta} \tag{14}$$

$$\hat{x}'_p(S) = \hat{x}_p(S) \cdot \exp\,[-u^*_{0.50}/\hat{\beta}] \tag{15}$$

5. PRECISION MEASURES

A useful index of how precisely β is determined by the series of life tests is the ratio R of the upper to lower ends of its confidence interval.

$$R = q_{(1-\alpha/2)}/q_{(\alpha/2)} \tag{16}$$

For γ the median length $L_{0.50}$ of the confidence interval is a convenient index calculated as

$$L_{0.50} = (w^*_{1-\alpha/2} - w^*_{\alpha/2})/\beta q_{0.50} \tag{17}$$

For $x_p(S)$ the median ratio $R_{0.50}$ of upper to lower confidence interval is recommended.

$$R_{0.50} = \exp\left[(-u^*_{\alpha/2} + u^*_{1-\alpha/2})/\beta q_{0.50}\right] \tag{18}$$

6. NUMERICAL EXAMPLE

Rolling contact fatigue tests of $n_i = 10$ hardened steel specimens were conducted at $k = 4$ levels of the contact stress. The tests were continued until all elements had failed ($r_i = 10$). The results are:

Stress 10^6-psi	Ordered Lives
0.87	1.67, 2.20, 2.51, 3.00, 3.90, 4.70, 7.53, 14.70, 27.76, 37.4
0.99	0.80, 1.00, 1.37, 2.25, 2.95, 3.70, 6.07, 6.65, 7.05, 7.37
1.09	0.012, 0.18, 0.20, 0.24, 0.26, 0.32, 0.32, 0.42, 0.44, 0.88
1.18	0.073, 0.098, 0.117, 0.135, 0.175, 0.262, 0.270, 0.350, 0.386, 0.456

The ML estimates are

$\hat{\beta} = 1.166$ $\qquad$ $\hat{\gamma} = 13.89$ $\qquad$ $\hat{\eta}_o = 2.20$

The 5-th, 50-th and 95-th percentiles of the distribution of q, w* and u* corresponding to $k = 4$, $n_i = r_i = 10$ and the specified values of S_i are given in Table 1. The distribution of u* was evaluated for $p = 0.10$ with $S = 0.75 \times 10^6$ psi and with the four test stresses.

TABLE 1

Percentiles of Pivotal Functions

$k = 4,\ n = r = 10$

$S_1=0.87,\ S_2=0.99,\ S_3=1.09,\ S_4=1.18$

	0.05	0.50	0.95
$q = \hat{\beta}/\beta$	0.8459	1.024	1.277
$w^* = (\hat{\gamma}-\gamma)\hat{\beta}$	-2.433	-0.3783	2.293
$u^* = \hat{\beta}\log[\hat{x}_{0.10}/x_{0.10}]$;			
$S = 0.75$	-0.9238	0.0555	1.023
$S = S_1 = 0.87$	-0.6495	0.0441	0.8209
$S = S_2 = 0.99$	-0.5170	0.0305	0.7520
$S = S_3 = 1.09$	-0.5309	0.0318	0.7671
$S = S_4 = 1.18$	-0.6079	0.0396	0.8193

90% confidence intervals for β and γ are

$$0.913 = 1.166/1.277 < \beta < 1.166/0.8459 = 1.378$$

$$11.92 = 13.889 - 2.293/1.166 < \gamma < 13.889 + 2.433/1.166 = 15.98$$

Median unbiased estimates are

$$\hat{\beta}' = 1.166/1.024 = 1.139$$

$$\hat{\gamma}' = 13.889 + 0.3783/1.166 = 14.21$$

The precision measures based on a 90% interval are

$R = 1.51$

$\beta L_{0.50} = 4.62$

The confidence intervals and unbiased estimates for $x_{0.10}$ are listed below for each stress along with the precision measure $R^{\beta}_{0.50}$.

Stress	Median Unbiased $\underline{x}_{0.10}$ Estimate	90% Confidence Interval $x_{0.10}$ Lower	Upper	$\underline{R}^{\beta}_{0.50}$
0.75	16.55	7.22	38.4	6.70
0.87	2.13	1.09	3.86	4.21
0.99	0.358	0.193	0.572	3.42
1.09	0.094	0.050	0.152	3.55
1.18	0.031	0.016	0.054	4.03

Figure 1 shows $R^{\beta}_{0.50}$ plotted against stress. The minimum value is only slightly larger then for a single uncensored sample of size $n = 40$.

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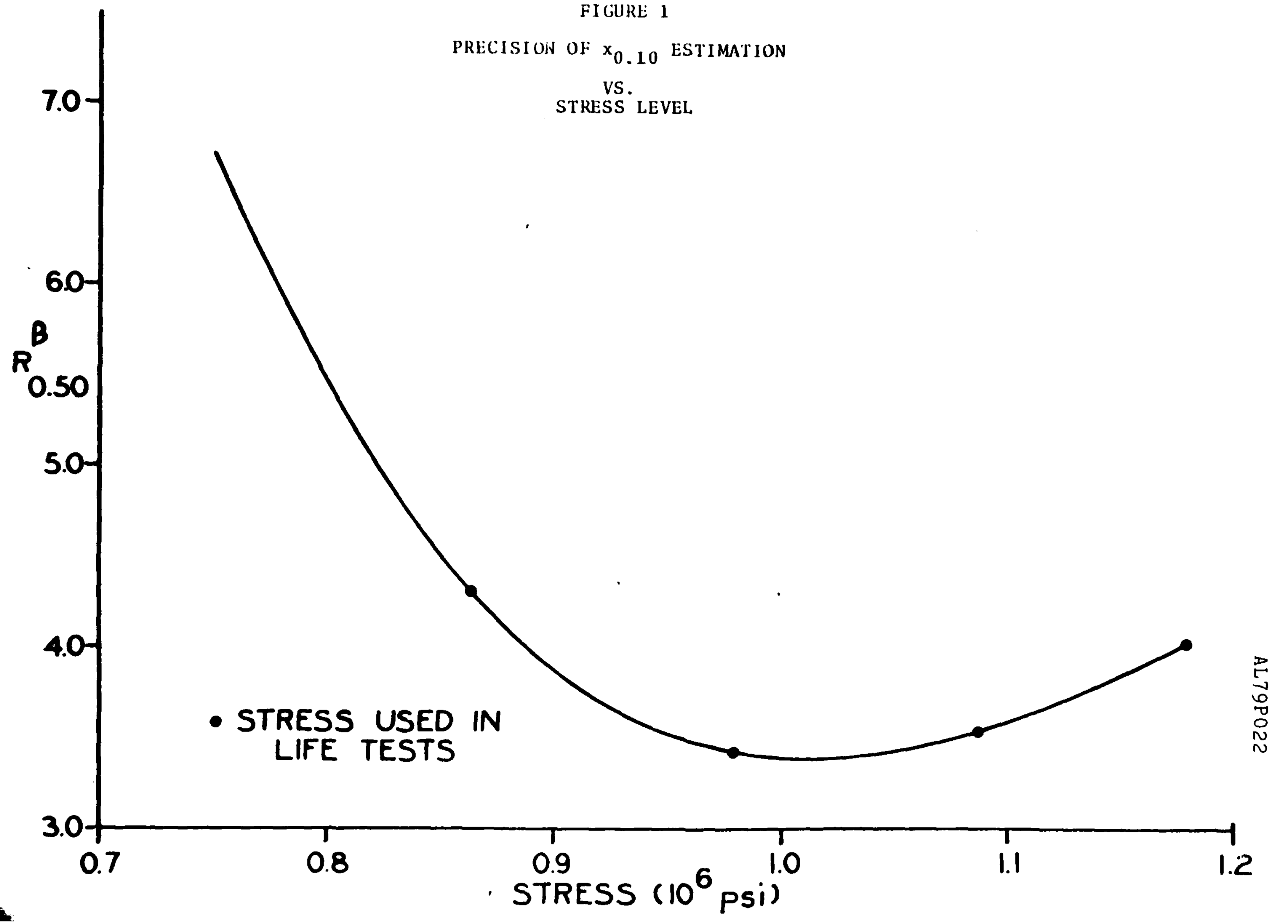
FIGURE 1

PRECISION OF $x_{0.10}$ ESTIMATION

VS.

STRESS LEVEL



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